Frequency Response

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Let a network be described by a transfer function $H(s)$. Let us denote the input signal as $r(t)$ and the output signal as $y(t)$. I am interested in finding the output $y(t)$ when the input is a sinusoidal function, i.e., $r(t) = A\sin(\omega_s t)$.

The Laplace transform of the input is:

$$\mathcal{L}[r(t)] = R(s) = \frac{A\omega_s}{s^2+\omega_s^2}$$

and hence the output can be written as:

$$Y(s) = H(s)R(s) = H(s)\frac{A\omega_s}{s^2+\omega_s^2}$$

Let

$$\frac{H(s)A\omega_s}{s^2+\omega_s^2} = \frac{k}{s+\jmath\omega_s} + \frac{k^*}{s-\jmath\omega_s} + \sum_{i=1}^{n}\frac{\lambda_i}{s+\alpha_i} \quad (1)$$

$$k = \left.\frac{H(s)A\omega_s}{s^2+\omega_s^2}(s+\jmath\omega_s)\right|_{s=-\jmath\omega_s} = \frac{H(-\jmath\omega_s)A\omega_s}{-2\jmath\omega_s} = \frac{\jmath H(-\jmath\omega_s)A}{2} \quad (2)$$

Let

$$H(\jmath\omega_s) \triangleq a(\omega_s) + \jmath b(\omega_s)$$

This implies:

$$Y_{ss}(s) \triangleq \frac{k}{s+\jmath\omega_s} + \frac{k^*}{s-\jmath\omega_s} = \frac{\jmath(a(\omega_s)-\jmath b(\omega_s))A}{2(s+\jmath\omega_s)} + \frac{-\jmath(a(\omega_s)+\jmath b(\omega_s))A}{2(s-\jmath\omega_s)}$$

Then:

$$\begin{aligned}
y_{ss}(t) &= \mathcal{L}^{-1}[Y_{ss}(s)] = \frac{(\jmath a(\omega_s)+b(\omega_s))A}{2}e^{-\jmath\omega_s t} + \frac{(-\jmath a(\omega_s)+b(\omega_s))A}{2}e^{\jmath\omega_s t} \\
&= A[\frac{\jmath a(\omega_s)}{2}(e^{-\jmath\omega_s t} - e^{\jmath\omega_s t})] + A[\frac{b(\omega_s)}{2}(e^{-\jmath\omega_s t} + e^{\jmath\omega_s t})] \\
&= A[\frac{a(\omega_s)}{2\jmath}(e^{\jmath\omega_s t} - e^{-\jmath\omega_s t})] + A[\frac{b(\omega_s)}{2}(e^{-\jmath\omega_s t} + e^{\jmath\omega_s t})] \\
&= A[a(\omega_s)\ \sin(\omega_s t) + b(\omega_s)\ \cos(\omega_s t)] \\
&= A\sqrt{(a^2(\omega_s)+b^2(\omega_s))}[\frac{a(\omega_s)}{\sqrt{(a^2(\omega_s)+b^2(\omega_s))}}\sin(\omega_s t) + \frac{b(\omega_s)}{\sqrt{(a^2(\omega_s)+b^2(\omega_s))}}\cos(\omega_s t)]
\end{aligned}$$

Noting that

$$|H(\jmath\omega_s)| = \sqrt{(a^2(\omega_s) + b^2(\omega_s))} \text{ and } \phi(\omega_s) \triangleq \angle H(\jmath\omega_s) = \tan^{-1}\frac{b(\omega_s)}{a(\omega_s)}$$

we can write:

$$y_{ss}(t) = A|H(\jmath\omega_s)|\sin(\omega_s t + \phi(\omega_s))$$

In the above we haven't paid any attention to the third part of the expression on the left hand side of the equation (1). Let us look at that expression now.

$$y_n(t) \triangleq \mathcal{L}^{-1}[\sum_{i=1}^{n}\frac{\lambda_i}{s+\alpha_i}] = \sum_{i=1}^{n}\lambda_i e^{-\alpha_i t}$$

Note that both λ_i and α_i can be either real or complex. When α_i is real then λ_i is also real. For complex α_i there will be another $\alpha_j = \alpha_i^*$ and in general we can write:

$$y_n(t) = \sum_{\forall \text{ real } \alpha_i} \lambda_i e^{-\alpha_i t} + \sum_{\forall \text{ complex } \alpha_i \ \& \ \alpha_i^*} k_i e^{-\Re(\alpha_i)t}\sin(\omega_s t + \psi)$$

It is easy to see that when (Real part of $(\alpha_i) > 0$) $\Re(\alpha_i) > 0$ then $\lim_{t\to\infty} y_n(t) = 0$.

For all passive RLC networks $\Re(\alpha_i)$ is always less than 0. The condition $\Re(\alpha_i) > 0$ also means that all the roots of the denominator of the transfer function $H(s)$ are in the left half of the complex plane. Roots of the denominator of the transfer function are also known as the system poles; zeros are the roots of the transfer function numerator.

In other words for any system with the poles in the left half complex plane the steady-state response to a sinusoid of frequency ω_s can be worked out by evaluating the magnitude and the phase of the complex number $H(\jmath\omega_s)$ and then noting that the magnitude of the output is given by the magnitude of the input sinusoid times the magnitude of $H(\jmath\omega_s)$ and the phase shift between the input sinusoid and the output is given by the phase of $H(\jmath\omega_s)$.