A Brief Introduction to Fourier Series using Digital Computers

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The most simple form of Fourier series can be written as:

$$f(\omega t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t \tag{1}$$

Keeping in mind that $\cos n\omega t$ and $\sin n\omega t$ are orthogonal functions with the following relationship (n, m integers):

$$\int_0^{2\pi} \sin n\omega t \cos m\omega t d\omega t = 0 \tag{2}$$

$$\int_0^{2\pi} \sin n\omega t \sin m\omega t d\omega t = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \tag{3}$$

$$\int_0^{2\pi} \cos n\omega t \cos m\omega t d\omega t = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \tag{4}$$

The expressions to obtain a_n and b_n are:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t) d\omega t \tag{5}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \cos n\omega t d\omega t, \quad n \neq 0 \tag{6}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \sin n\omega t d\omega t, \quad n \neq 0 \tag{7}$$

Note that equation (1) can also be represented as:

$$f(\omega t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(n\omega t - \tan^{-1}(b_n/a_n)) \tag{8}$$

The above Fourier series given by equation (8) can also be written as:

$$f(\omega t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} \tag{9}$$

where C_n are the complex coefficients given by:

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t) e^{-jn\omega t} d\omega t \tag{10}$$

Note that $|C_n|$ gives the magnitude of the n^{th} harmonic in the signal. It's very useful to get a digital computer to work out the coefficients C_n for a given time signal. In practice a time waveform is recorded in a digital computer by time sampling the waveform. This means that instead of the entire function $f(\omega t)$ we know the function only for a few selected time instances, i.e., we store the samples, $f(m/P), \ \ m = 1, 2, \ldots, N$ where P is the number of samples collected per second and N is the total number of samples. To keep the notation simple we denote samples as $f(k)$ where the unit of k is seconds and equals m/P.

Since a digital computer doesn't have the entire time function it cannot perform the integration as required by equation (10). Instead a summation is performed and the integral is approximated as shown below:

$$C_n = \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-j\frac{2\pi}{N}nk}, \quad n = 0, \ldots, N-1 \tag{11}$$

The original signal can be represented as:

$$f(k) = \sum_{n=0}^{N-1} C_n e^{j\frac{2\pi}{N}nk}, \quad k = 0, \ldots, N-1 \tag{12}$$

From equation (11) it can be seen that:

$$\begin{aligned} C_{N-n} &= \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-j\frac{2\pi}{N}(N-n)k} && (13) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-j2\pi k} e^{j\frac{2\pi}{N}nk} && (14) \\ &= C_n^* \text{ when } f(k) \text{ real.} && (15) \end{aligned}$$

Let us define $C_n = a_n + jb_n$ (a_n, b_n different from the ones in equation (1)), then $C_{N-n} = a_n - jb_n$.

Coming back to the equation (12) we see that (let N be even):

$$\begin{aligned} f(k) &= C_0 + \sum_{n=1}^{\frac{N}{2}-1} C_n e^{j\frac{2\pi}{N}nk} + C_{\frac{N}{2}} e^{j\pi k} + \sum_{n=\frac{N}{2}+1}^{N-1} C_n e^{j\frac{2\pi}{N}nk}, \quad k = 0, \ldots, N-1 \\ &= C_0 + \sum_{n=1}^{\frac{N}{2}-1} C_n e^{j\frac{2\pi}{N}nk} + C_{\frac{N}{2}} e^{j\pi k} + \sum_{n=1}^{\frac{N}{2}-1} C_{N-n} e^{j\frac{2\pi}{N}(N-n)k} \\ &= C_0 + \sum_{n=1}^{\frac{N}{2}-1} (a_n + jb_n) e^{j\frac{2\pi}{N}nk} + C_{\frac{N}{2}} e^{j\pi k} + \sum_{n=1}^{\frac{N}{2}-1} (a_n - jb_n) e^{-j\frac{2\pi}{N}nk} \\ &= C_0 + \sum_{n=1}^{\frac{N}{2}-1} 2(a_n \sin \frac{2\pi}{N}kn + b_n \cos \frac{2\pi}{N}kn + C_{\frac{N}{2}} e^{j\pi k} \\ &= C_0 + \sum_{n=1}^{\frac{N}{2}-1} 2\sqrt{a_n^2 + b_n^2} \cos(\frac{2\pi}{N}kn + \tan^{-1}(b_n/a_n)) + C_{\frac{N}{2}} e^{j\pi k} && (16) \end{aligned}$$

It can be seen from equation (16) that the time function $f(k)$ is a summation of various frequency components. The fundamental component has a period of N samples; $f(k) = f(k+N)$. This in real terms means that the signal has a period of $\frac{N}{P}$ seconds (P is the number of samples per second); the fundamental frequency is $\frac{P}{N}$ Hz and the highest frequency component is $\frac{P}{N} \times \frac{N}{2} = \frac{P}{2}$ Hz. In other words the resolution of an N point Fourier series (P samples per second) is $\frac{P}{N}$ Hz. In the literature the Fourier series given by the equation (12) is called discrete Fourier transform (DFT) and those algorithms which calculate this DFT using some special algorithm are called Fast Fourier Transforms (FFT).