Linear Differential Equations and Their Solution

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1 Introduction

This short notes tries to demystify the process of obtaining solutions of linear differential equations.

The first thing to understand is that given any big or small problem the solution process starts with guessing a solution and then trying out if the guess is indeed a solution. Most researchers hide their intial guess $\rightarrow$ verify efforts when they publish their work. There is a desire to present a polished work. But this is detrimental to the learning value of the published work.

In learning and teaching, the process of guessing a solution, trying it out, and then successively improving the guess should be followed as much as is practical.

In most fields of study, a general method is specialised in different ways to solve specific problems. The specialisation is done to economise on the efforts to obtain solutions. In the learning process it's counterproductive to spend too much time on short-cuts. Instead all the effort should be first concentrated on fully grasping the general approach. The easy availability of computers should be used as an aid in learning. A general solution method, once fully grasped, can be easily programmed on a computer. Thus the mechanical steps in solving a problem are eliminated. This makes it pointless to spend time and efforts in learning short-cuts.

Once the fundamentals are grasped, specialisation is always welcome. But a clear distinction between general and special should be always maintained for a learner.

This idea can be illustrated in learning how to obtain solutions of linear differential equations. We use the guess $\rightarrow$ verify $\rightarrow$ improve method to fully grasp the general solution method of linear differential equations.

2 First order Linear Differential Equation

Let us look at the following simple first order linear differential equation:

$$\dot{x} = ax, \;\; x(0) = x_0 \tag{1}$$

The solution of this equation (1) is a function which when differentiated is a constant times the function itself. Which function is that? The exponential function has this property,

$$\frac{de^{at}}{dt} = ae^{at}. \tag{2}$$

So $x(t) = e^{at}$ satisfies the differential equation (1) but not the initial condition. We need to modify our guess of $x(t)$ only slightly to satisfy the initial condition: $x(t) = x_0 e^{at}$. This solution is known as the initial condition solution.

Now we look at the forced first order equation:

$$\dot{x} = ax + bu(t), \quad x(0) = x_0 \tag{3}$$

where $u(t)$ is the forcing function.

As a first guess let's see how our initial condition solution can help us here. Let the guess be $x(t) = x_0 e^{at} + y(t)$, $y(t)$ is as yet unknown function. It would make sense to impose the condition $y(0) = 0$. Substituting this guess in equation (3) gives,

$$\begin{aligned} ax_0 e^{at} + \dot{y}(t) &= a\left(x_0 e^{at} + y(t)\right) + bu(t) \\ &\Rightarrow \dot{y}(t) = ay(t) + bu(t) \end{aligned} \tag{4}$$

Equation (4) is similar to the original equation but with zero intial condition.

As we did for the solution of the unforced equation (1), we guess a solution for the forced equation (4) as $y(t) = e^{at} z(t)$, where $z(t)$ is as yet unknown function. Substituting this guess in equation (4) we get,

$$\dot{y}(t) = ae^{at} z(t) + e^{at} \dot{z}(t) \tag{5}$$

From equation (5) we can see that if

$$\dot{z}(t) = e^{-at} bu(t) \tag{6}$$

then $y(t) = e^{at} z(t)$ is a solution of equation (4). This means that the solution of the original equation (3) is,

$$x(t) = x_0 e^{at} + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau. \tag{7}$$

3 Second Order Linear Differential Equation

Now we look at the solution of the following homogeneous second order linear differential equation.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0, \quad x(0) = x_0, \; x'(0) = x'_0 \tag{8}$$

3.1 Solution

To find a solution of the above differential equation (8), we first guess a solution and then substitute it back in the equation and see if it satisfies the equation. This process is very similar to what we do to find roots of an algebraic equation. To see what type of functions may provide a good guess we rewrite the above equation as: $\frac{d^2x}{dt^2} = -a\frac{dx}{dt} - bx$, and observe that higher order derivatives are expressed as a linear combination of the lower order derivatives. Which functions have this property? Polynomial functions in time don't have this property (why?). Sine and cosine functions differentiated twice result in a constant times the function itself, so sine and cosine functions show some promise and they are indeed good guesses for equations where the first derivative is missing, i.e., $\frac{d^2x}{dt^2} = -bx$, but for a general equation (8) it can't be used.

Drill Substitute a general function $A\sin(\lambda x + \phi)$ in equation (8) and show that such a function can't satisfy a general second order differential equation.

Since the sine function shows some promise let's go a step further and try a generalised trigonometric function, $ke^{\lambda x}$. On substituting $ke^{\lambda x}$ into equation (8) we get,

$$\lambda^2 ke^{\lambda x} + a\lambda ke^{\lambda x} + bke^{\lambda x} = 0. \tag{9}$$

For the above equation to hold we need,

$$\lambda^2 + a\lambda + b = 0 \tag{10}$$

Equation (10) is known as the characteristic equation and let its solution be λ_1 and λ_2. Then the general solution of the differential equation (8) is given by:

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \tag{11}$$

3.1.1 Real Roots (Overdamped Solution)

The case where λ_1 and λ_2 are real and distinct needs no further comment except that the initial conditions are used to find the values of the constants k_1 and k_2. The case with equal real roots is interesting and is covered later.

3.1.2 Complex Roots (Underdamped Solution)

When λ_1 and λ_2 are complex they have to be complex conjugate (why?). This means that, $\lambda_1, \lambda_2 = \alpha \pm \jmath\omega$. The general solution can then be written as:

$$\begin{aligned} x(t) &= k_1 e^{\alpha + \jmath\omega t} + k_2 e^{\alpha - \jmath\omega t} \\ &= e^{\alpha t}\left((k_1 + k_2)\cos(\omega t) + \jmath(k_1 - k_2)\sin(\omega t)\right) \end{aligned} \tag{12}$$

To satisfy the given initial conditions it must be true that,

$$k_1 + k_2 = x(0) \tag{13}$$

$$\jmath\omega(k_1 - k_2) + \alpha(k_1 + k_2) = x'(0) \tag{14}$$

From (13) and (14) above, for real values of $x(0)$ and $x'(0)$, k_1 and k_2 have to be complex conjugate.

Drill Prove that k_1 and k_2 are complex conjugate.

For complex conjugate k_1 and k_2 we see that $\jmath(k_1 - k_2)$ is a real number. Denote:

$$\begin{aligned} k &= \sqrt{(k_1 + k_2)^2 + (\jmath(k_1 - k_2))^2} = 2k_1k_2 \\ \phi &= \tan^{-1}\left(\frac{k_1 + k_2}{\jmath(k_1 - k_2)}\right) \end{aligned}$$

with the above definitions of k and ϕ the general solution (12) can be written in the form:

$$x(t) = ke^{\alpha t}\sin(\omega t + \phi) \tag{15}$$

The two constants k and ϕ are solved for the given initial conditions.

3.1.3 Repeated Roots (Critically Damped Solution)

When the roots of the characteristic equation (10) are equal then the solution is written differently from (11) above. The characteristic equation such as (10) has repeated roots when $b = a^2/4$ and then the differential equation (10) can be written as:

$$\frac{d}{dt}\left(\frac{dx}{dt} + \frac{a}{2}x\right) + \frac{a}{2}\left(\frac{dx}{dt} + \frac{a}{2}x\right) = 0 \tag{16}$$

Let

$$y \triangleq \frac{dx}{dt} + \frac{a}{2}x \tag{17}$$

with this definition of y, equation (16) and its solution can be written as:

$$\frac{dy}{dt} + \frac{a}{2}y = 0 \Rightarrow y = ke^{-\frac{a}{2}t}. \tag{18}$$

Substituting the above solution back into (17) we get:

$$\frac{dx}{dt} + \frac{a}{2}x = ke^{-\frac{a}{2}t} \Rightarrow x = k_1te^{-\frac{a}{2}t} + k_2e^{-\frac{a}{2}t} \tag{19}$$

Like the previous two cases the initial conditions are used to obtain values of the constants k_1 and k_2.

3.2 Forcing Functions

For an equation with a forcing function $f(t)$, shown below,

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = f(t), \quad x(0) = x_0, \; x'(0) = x_0', \tag{20}$$

what has been described before is used to get the natural response.

To find particular solution for a general forcing function is a bit tedious. Particular solutions can be easily obtained for certain forcing functions. For example, particular solution due to a step forcing function is a constant, and due to a sinusoidal forcing function is a sinusoid itself. Its frequency is the same as that of the forcing function sinusoid but its magnitude and phase are different.

Earlier we saw that it was very simple to obtain a complete solution to the first order equation (3). In the next section we use the same idea to obtain solution of simultaneous first order differential equations.

4 Simultaneous First Order Differential Equations

The second order equation (20) can be written as two simultaneous first order equations. Define

$$x_1 \;:=\; x \tag{21}$$

$$x_2 \;:=\; \dot{x} \Rightarrow \dot{x}_2 = -ax_2 - bx_1 + f(t) \tag{22}$$

With the above definitions (21) and (22), the differential equation (20) can be written in compact matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t) \tag{23}$$

We now write the above equation in the form,

$$\dot{X} = AX + Bf(t) \tag{24}$$

where

$$X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \;\; A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{25}$$

We say that equation (24) is a vector version of equation (3). Looking at the solution of equation (3) given in equation (7) we write the solution of equation (24) as:

$$X(t) = \Phi(t)\left[X(0) + \int_0^t \Phi(-\tau)Bf(\tau)d\tau\right]. \tag{26}$$

The solution (26) is valid provided function $\Phi(t)$ satisfies two conditions:

$$\dot{\Phi}(t) = A\Phi(t) \text{ and } \Phi(t)\Phi(-t) = I$$

where I is an identity matrix. Note that

$$\frac{d}{dt}\int_0^t \Phi(-\tau)Bf(\tau)d\tau \ = \Phi(-t)Bf(t). \tag{27}$$

If $\dot{\Phi}(t) = A\Phi(t)$ then $\ddot{\Phi}(t) = A^2\Phi(t)$, etc. We can use this fact to form a Taylor series for the function $\Phi(t)$ (note that $\Phi(0) = I$, $\dot{\Phi}(0) = A$, and $\ddot{\Phi}(0) = A^2$, etc.)

$$\Phi(t) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \tag{28}$$

It can be verified by direct evaluation that

$$\Phi(t)\Phi(-t) = I \text{ and } \Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2).$$

There exist many different ways to calculate the state-transition matrix $\Phi(t)$. One of the easier ways is to use a computer. The factorial terms in equation (28) dominate after a few terms for small t. This with the property $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$ makes it very easy to calculate state-transition matrix using a computer.

Simultaneous first order differential equations for an nth order equation can also be written quite easily. Let the nth order system be given as:

$$\frac{d^nx}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_nx = f(t), \quad \frac{dx^i(0)}{dt^i} = x_0^i, \quad i = 0, 2, \dots, n-1. \tag{29}$$

Then define:

$$\begin{aligned} x_1 &:= x \\ x_i &:= \dot{x}_{i-1}, \quad i = 2, \dots, n \end{aligned} \tag{30}$$

This gives,

$$\dot{x}_n = -\sum_{i=1}^{n} a_i x_{n-i+1} + f(t). \tag{31}$$

The n first order simultaneous equations (30) and (31) can be written as:

$$\dot{X} = AX + Bf(t), \quad X(0) = X_0 \tag{32}$$

where

$$X := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad X_0 := \begin{bmatrix} x_0^0 \\ x_0^1 \\ \vdots \\ x_0^{n-2} \\ x_0^{n-1} \end{bmatrix}. \tag{33}$$

The solution of equation (32) is:

$$X(t) = \Phi(t)\left[X(0) + \int_0^t \Phi(-\tau)Bf(\tau)d\tau\right]. \tag{34}$$

This is all you need for any linear differential equation. A general program can be written to solve any differential equation without worrying about various short-cuts.